Tracking in Clutter With Nearest Neighbor Filters: Analysis and Performance

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The measurement that is "closest" to the predicted target measurement is known as the "nearest neighbor" (NN) measurement in tracking. A common method currently in wide use for tracking in clutter is the so-called NN filter, which uses only the NN measurement as if it were the true one. The purpose of this work is two fold. First, the following theoretical results are derived: the a priori probabilities of all three data association events (updates with correct measurement, with incorrect measurement, and no update), the probability density functions (pdfs) of the NN measurement conditioned on the association events, and the one-step-ahead prediction of the matrix mean square error (MSE) conditioned on the association events. Secondly, a technique for prediction *without recourse to expensive Monte Carlo simulations* of the performance of tracking in clutter with the NN filter is presented. It can quantify the dynamic process of tracking divergence as well as the steady-state performance. The technique is a new development along the line of the recently developed general approach to the performance prediction of algorithms with both continuous and discrete uncertainties.

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I. INTRODUCTION

The measurement that is "closest" to the predicted target-originated measurement is known as the "nearest neighbor" (NN) measurement in target tracking. One of the most common and widely used methods for tracking in clutter is the so-called NN filter which uses at any time only the NNmeasurement *as if it were* the one originated from the target of interest [3, 4]. This filter can be used either alone or as a module in a more complex algorithm.

Efforts have been made for the performance analysis of the NN filter. The pioneering work of Sea and Singer was documented in e.g., [17, 18]. Recent advances include [16], where the tracking error is modeled as a diffusion process. Very convenient and revealing analytic relationships between performance measures and scenario parameters have been established.[1] On the other hand, however, the following important scenario-dependence of the filter performance has been overlooked: The performance of the filter depends to a large extent on the scenario in which it is running, not just *scenario parameters*. As a result, a quantitative prediction of the filter performance, especially the dynamic process of the important *tracking divergence*, is not available in the literature.

Based on the Hybrid Conditional Averaging (HYCA) technique developed recently by the authors [10, 11, 13], we present a technique for quantitative evaluation (prediction) of the performance of the NN filter without recourse to costly and time-consuming Monte Carlo simulations. This technique for the first time treats the performance of the NN filter for tracking in clutter as a bona fide dynamic process. This is achieved by carefully evaluating the evolution of the estimation errors under all three possible data association events: 1) the NN measurement is a false one; 2) the NN measurement is the correct one; and 3) there is no NN measurement. The probabilities of these three events as well as the probability density functions (PDFs) of the nearest neighbor measurement conditioned on the last two events[2] are also presented. These conditional pdfs are essential for the derivation of the evolution of the average estimation errors. The association probabilities are used as the weights for the corresponding estimation errors. They can also be used for other purposes to be illustrated in a forthcoming paper.

This work, which is a combination and refinement of two recent conference papers [8, 12], is organized as follows. Section II provides a brief description of the NN filter, together with the validation gate. The theoretical results of the conditional pdfs of the NN

[1] A relevant progress is the recent work of [14], where an approximate analysis of the data association with the scan-wise optimal assignment algorithm is presented.

[2] The pdfs have been incorrectly calculated/assumed prior to [8].

measurement and the probabilities of data association events are presented in Section III. In Section IV, the propagation of the average tracking error of the NN filter is obtained. This leads to an off-line recursive performance predictor. The accuracy of this performance predictor is demonstrated via a numerical example in Section V. Section VI concludes with a summary. Mathematical details are given in the Appendix.

II. NN FILTER

Let z denote the n-dimensional measurement vector of the target state. Note that it is possible to have more than one measurement at any time even for single target tracking because a measurement may have originated from a source other than the target of interest. Let $Z(k)$ be the set of validated measurements (to be defined later) at time k ($Z(k)$ could be an empty set). Denote the validated measurement *set sequence* through time k as Z^k, i.e., $Z^k = \{Z(0), Z(1), \ldots, Z(k)\}$, (or more rigorously, the corresponding σ-algebra), where $Z(0)$ denotes the system structure information and initial conditions for the estimation. Let $z_*(k)$ be the NN measurement at time k, defined by

$$z_*(k) = \arg \min_{z \in Z(k)} D(z) \tag{1}$$

where the *normalized distance squared* (NDS) $D(z)$ is defined as

$$D(z) = [z - \hat{z}(k \mid k-1)]' S(k)^{-1} [z - \hat{z}(k \mid k-1)] \tag{2}$$

in which $\hat{z}(k \mid k-1)$ is the predicted measurement and $S(k)$ is its associated covariance matrix, given later.

An important concept in target tracking is that of a validation gate. Loosely speaking, a validation gate is such a region in the measurement space that a measurement falling outside this region is deemed to be generated from a source other than the target of interest. The most commonly used validation gate is the ellipsoid:

$$R_\gamma = \{z : D(z) \le \gamma\} \tag{3}$$

where $g = \sqrt{\gamma}$ is called the *gate size*. A measurement which falls inside the validation gate is referred to as a *validated measurement*. The volume of the n-dimensional gate R_γ is

$$V_\gamma(k) = c_n |S(k)|^{1/2} \gamma^{n/2} \tag{4}$$

where $|\cdot|$ denotes determinant and c_n is the volume of the n-dimensional unit hypersphere, given by

$$c_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}. \tag{5}$$

Similarly, for the convenience of reference, an ellipsoid of an arbitrary size $\sqrt{D}$ is denoted by

$$R_D = \{z : D(z) \le D\} \tag{6}$$

and its volume

$$V_D(k) = c_n |S(k)|^{1/2} D^{n/2}. \tag{7}$$

Consider the following target motion model

$$x(k) = F(k-1)x(k-1) + G(k-1)v(k-1) \tag{8}$$

with target measurements

$$z_T(k) = H(k)x(k) + w(k) \tag{9}$$

where $x \in R^{n_x}$ is the target state vector; $z_T \in R^n$ is the noisy measurement vector; $v(k-1)$ is the Gaussian white process noise sequence with mean $\bar{v}(k-1)$ and covariance $Q(k-1)$; and $w(k)$ is the zero-mean Gaussian white measurement noise sequence with covariance $R(k)$. It is assumed that $v(k)$, $w(k)$, and $x(0)$ are uncorrelated. The system structure matrices F, G, and H and the noise statistics $\bar{v}$, Q, and R are assumed known to the filter.

Based on the Kalman filtering theory, the NN filter makes the following prediction [2]:

$$\hat{x}(k \mid k-1) = F(k-1)\hat{x}(k-1 \mid k-1) + G(k-1)\bar{v}(k-1) \tag{10}$$

$$\hat{z}(k \mid k-1) = H(k)\hat{x}(k \mid k-1) \tag{11}$$

$$P(k \mid k-1) = F(k-1)P(k-1 \mid k-1)F(k-1)' + G(k-1)Q(k-1)G(k-1)'. \tag{12}$$

In the update the NN filter uses the NN measurement as if it were from the target of interest:

$$\hat{x}(k \mid k) = \hat{x}(k \mid k-1) + W(k)\nu(k) \tag{13}$$

$$P(k \mid k) = P(k \mid k-1) - W(k)S(k)W(k)' \tag{14}$$

where

$$\nu(k) = z_*(k) - \hat{z}(k \mid k-1) \quad \text{(measurement residual)} \tag{15}$$

$$S(k) = H(k)P(k \mid k-1)H(k)' + R(k) \quad \text{(measurement residual covariance)} \tag{16}$$

$$W(k) = P(k \mid k-1)H(k)'S(k)^{-1} \quad \text{(filter gain)} \tag{17}$$

and if there is no validated measurement, then

$$\hat{x}(k \mid k) = \hat{x}(k \mid k-1) \tag{18}$$

$$P(k \mid k) = P(k \mid k-1). \tag{19}$$

III. PDF OF NN MEASUREMENT AND DATA ASSOCIATION PROBABILITIES

In a cluttered environment, three events concerning the NN measurement may occur at any time:

Event M_0: There is no validated measurement and thus no NN measurement.

Event M_T: The NN measurement is from the target of interest.

Event M_F: The NN measurement is not from the target of interest.

For clarity, the common assumptions used here are listed below.

A1. The detection of the true target is independent of false detections once the detection threshold of the tracking sensor is determined.

A2. The number of validated target-originated measurements, denoted by m_T, is a binary random variable. Specifically, it is assumed that the conditional probability (probability mass function (pmf)) at any time k is, with Assumption A1,

$$P\{m_T(k) \mid Z^{k-1}\} = P_D P_G \delta[m_T(k) - 1] + (1 - P_D P_G)\delta[m_T(k)] \quad (20)$$

where $\delta[\cdot]$ is the Kronecker delta function, P_D is the probability of target detection, and P_G is the gate probability, that is, the probability that the target-originated measurement falls inside the gate assuming that the target is detected.

A3. The target-originated measurement, denoted by z_T, is Gaussian distributed with an appropriate mean and covariance, that is, $z_T(k) \sim \mathcal{N}[\hat{z}(k \mid k-1), S(k)]$. Note that Assumption A3 is accurate only when $\hat{z}(k \mid k-1)$ and $S(k)$ are accurate. As time goes, it will become less and less accurate if the filter is diverging.

A4. The number of false validated measurements in any region of the gate at any time k, denoted by $m_F(k)$, can be described by a suitable Poisson model with a certain spatial density λ. For example, the probability of the total number of false measurements in the gate is given by

$$P\{m_F(k) = m\} = \frac{(\lambda V_\gamma)^m}{m!} e^{-\lambda V_\gamma}. \quad (21)$$

A5. The false measurements at any time are independent identically distributed (i.i.d.) uniformly distributed over the validation gate.

A6. The location of a false measurement inside the gate is independent of the target-originated measurement and of false measurements at any other times.

These assumptions are commonly made for target tracking in clutter [3]. Based on these assumptions, the conditional pdfs of the NN measurement under M_T and M_F, together with the correct and incorrect data association probabilities as well as the probability of no measurement, are obtained below.

A. PDF of NN Measurement When it is Target Originated

In [6, 16], the pdf of $z_*(k)$, assuming it is originated from the target, was given by (conditioning on Z^{k-1} is omitted for brevity)[3]

$$p_{z_*}[y \mid M_T(k)] = P_G^{-1} \mathcal{N}[y; \hat{z}(k \mid k-1), S(k)] U(y; R_\gamma) \quad (22)$$

where $\mathcal{N}[y; \hat{z}(k \mid k-1), S(k)]$ is a multivariate Gaussian pdf with the specified mean and covariance; and $U(y; R_\gamma)$ is a multivariate unit step function, defined as

$$U(y; R_\gamma) = \begin{cases} 1 & y \in R_\gamma \\ 0 & \text{elsewhere} \end{cases}. \quad (23)$$

However, this is, in general, incorrect due to the possible existence of the false measurement(s) in the gate at time k. Actually, (22) is valid only under the condition that there is no false measurement in the gate at k. To see this intuitively, denote by $z_F(k)$ the *false* measurement *closest* to $\hat{z}(k \mid k-1)$ with $D(z_F) = D_F \leq \gamma$. Clearly, this imposes a constraint via the NN criterion on the region in which $z_*(k) = z_T(k)$ can lie. In other words, in order for $M_T(k)$ to be true, the valid range for $z_T(k)$ now is $\{z : D(z) \leq D_F\}$ rather than R_γ as in the case of no false measurements.

THEOREM 1 *With Assumptions A1–A4, the pdf of $z_*(k)$, conditioned on Z^{k-1} (omitted below) and $M_T(k)$, is given by*

$$p_{z_*}[y \mid M_T(k)] = \frac{P_D}{P\{M_T(k)\}} e^{-\lambda V_{D(y)}(k)} \times \mathcal{N}[y; \hat{z}(k \mid k-1), S(k)] U(y; R_\gamma) \quad (24)$$

where $P\{M_T(k)\}$, to be given later, is the probability of the correct data association, that is, the probability that the target-originated measurement is the NN measurement at time k; $V_{D(y)}$ is the volume of the gate with gate size $\sqrt{D(y)}$, defined in (7)*; and λ is the spatial density of the false measurements.*

PROOF See Appendix.

REMARKS 1) The pdf described in (24) is elliptically symmetric[4] [5] since

$$p_{z_*}(y \mid M_T) = p[D(y) \mid M_T] \quad (25)$$

for $D(\cdot)$ defined by (2) with some $S(k)$. 2) The PDF given in (24) is in general not Gaussian except for the two-dimensional case ($n = 2$), in which case

$$p[y \mid M_T] = \frac{P_D}{P\{M_T\} |2\pi S|^{1/2}} \times \exp\{-[\tfrac{1}{2} + \lambda c_n |S|^{1/2}] D(y)\} U(y; R_\gamma) \quad (26)$$

[3]It was assumed in [16] that $P_G = 1$.

[4]Elliptically symmetric distributions are multivariate distributions whose equi-probability contours are ellipsoids.

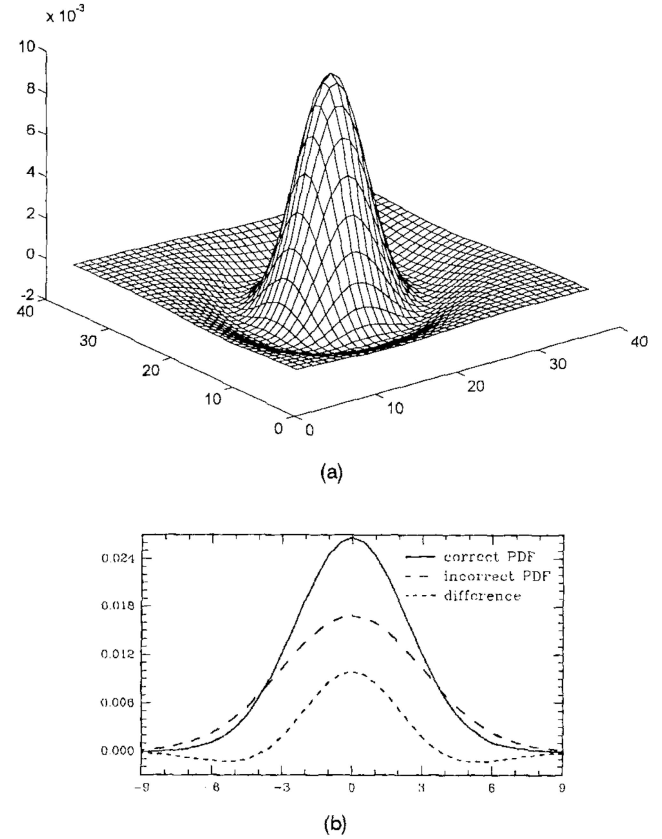

Fig. 1. Error of (22) for typical 2-dimensional tracking scenario. (a) 3-dimensional view of error. (b) Vertical cross section passing through vertical axis.

is a truncated Gaussian pdf over the gate, but with a covariance different from that of (22).

Comparing (26) and (22), it is clear that (22) is not a good approximation of (26) unless $\beta = \lambda c_n |S(k)|^{1/2}$, the expected number of the false measurements in a 1-σ gate, is negligible relative to 1/2, which does not hold even for a moderate clutter density. Note that the expected number β is negligible if 1) the spatial density λ of the false measurements is very low, meaning that there are hardly any false measurements at any time and/or 2) $|S(k)|^{1/2}$ is very small, meaning that the Gaussian random variable $z_T(k)$ concentrates very much around its mean (the center of the gate). This agrees with the intuition. Fig. 1 illustrates the difference between the pdfs obtained from (26) and from (22) for the following typical set of values:

$$P_D = 0.7, \qquad P_G = 0.99, \qquad \lambda = 0.01, \qquad S = \begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix} \tag{27}$$

which leads to $\beta = 0.3$.

It is observed that the error of (22) exceeds 30% of the peak value of the pdf in the vicinity of the gate center. Note that a small error in the pdf, especially around the peak, will lead to a large error in the cumulative distribution function (cdf).

In general, the pdf is a Gaussian density scaled by an exponentially decreasing factor $e^{-\lambda V_D}$, together with an associated constant. The error of (22) has a general pattern similar to that shown in Fig. 1. The constant factor is essentially responsible for the large error around the center; and the error in the ring region (elliptical layer, in the general case) is mainly due to the exponential factor. It can be expected that the higher the dimension is, the larger the error of (22) is.

The pdf of the NDS $D_*(k)$ associated with the NN measurement is also of interest. Owing to the fact that the conditional pdfs of the NN measurement under M_T and M_F (to be given later), respectively, are elliptically symmetric, the following proposition will prove to be very useful.

PROPOSITION 1 *If a pdf $p_z(z)$ is elliptically symmetric, that is,*

$$p_z(z) = p_z[D(z)]U(z;R_\gamma) \tag{28}$$

for $D(\cdot)$ defined by (2) *with some $S(k)$, then the following holds for the pdf of D:*

$$p_D(D) = \frac{nV_D}{2D} p[D(z)]U(D;(0,\gamma]) \tag{29}$$

where V_D is defined by (7). *The converse of the above is also true.*

PROOF See Appendix.

COROLLARY 1.1 *Under Assumptions A1–A4, the pdf of the NDS $D_*(k)$ associated with the NN measurement, assuming it is from the target, is given by*

$$p_{D_*}[D \mid M_T(k)] = \frac{D^{n/2-1}e^{-D/2}}{2^{n/2}\Gamma(n/2)} \frac{P_D}{P\{M_T(k)\}} \times e^{-\lambda V_D(k)} U(D;(0,\gamma]) \tag{30}$$

or

$$p_{D_*}[D \mid M_T(k)] = \frac{nV_D(k)}{2D} \frac{P_D}{P\{M_T(k)\}} \times e^{-\lambda V_D(k)} \frac{e^{-D/2}}{|2\pi S(k)|^{1/2}} U(D;(0,\gamma]). \tag{31}$$

PROOF See Appendix.

REMARK The Gaussian assumption (A3) for z_T implies that $D(z_T)$ without conditioning on M_T has a central chi-square pdf $p_{\chi^2(n)}(D)$ of n degrees of freedom which is the first fraction in (30). The remaining factor in (30) reveals that the possible existence of the false measurements has an impact as discussed above.

B. PDF of NN Measurement When it is not Target-Originated

The pdf of $z_*(k)$, assuming it is *not* from the target of interest, was also calculated/assumed incorrectly in [6, 16] because the effects of the possible target-originated measurement and other false measurements were not accounted for. The following proposition quantifies the effect of more than one false measurement.

PROPOSITION 2 *Let $z_F(k)$ be the false validated measurement closest to $\hat{z}(k \mid k-1)$. Under Assumptions A4–A5, the pdf of $z_F(k)$ is given by*

$$p_{z_F(k)}(y) = \frac{\lambda e^{-\lambda V_{D(y)}}}{1 - e^{-\lambda V_\gamma}} U(y; R_\gamma). \tag{32}$$

PROOF See Appendix.

Similarly to Corollary 1.1, the following lemma presents the corresponding pdf of the smallest NDS D_* under M_F.

LEMMA 1 *With Assumptions A1–A6, the pdf of the NDS D_* under the condition that the NN measurement is not from the target of interest is given by*

$$p_{D_*}[D \mid M_F(k)] = \frac{nV_D}{2D} \frac{\lambda e^{-\lambda V_D}}{P\{M_F(k)\}} \times [1 - P_D P\{\chi^2(n) \le D\}] U(D; (0, \gamma]). \tag{33}$$

PROOF See Appendix.

THEOREM 2 *Under Assumptions A1–A6, the pdf of the NN measurement, assuming it is not from the target of interest, is given by*

$$p_{z_*}[y \mid M_F(k)] = \frac{\lambda e^{-\lambda V_{D(y)}}}{P\{M_F(k)\}} \times [1 - P_D P\{\chi^2(n) \le D(y)\}] U(y; R_\gamma) \tag{34}$$

where $P\{M_F(k)\}$, to be given later, is the incorrect data association probability, that is, the probability that a false measurement is the NN measurement in the NN filter at time k; and $\chi^2(n)$ denotes a random variable with a central chi-square distribution of n degrees of freedom and thus

$$P\{\chi^2(n) \le D(y)\} \triangleq \int_0^{D(y)} \frac{q^{n/2-1} e^{-q/2}}{2^{n/2}\Gamma(n/2)} dq. \tag{35}$$

PROOF See Appendix.

REMARKS 1) The pdf of (34) is also elliptically symmetric. 2) Comparing (34) and (32), it is observed that the term $(1 - e^{-\lambda V_\gamma})[1 - P_D P\{\chi^2(n) \le D(y)\}]/P\{M_F(k)\}$ accounts exactly for the effect of the possible existence of the target-originated measurement. Specifically, the closest false validated measurement must have a distance squared smaller than that of the target-originated one (which is a $\chi^2(n)$ random variable), if it is detected. 3) The difference between (32) and V_γ^{-1}, the (uniform) pdf of a single false validated measurement, reflects exactly the effect of the other false validated measurements on the pdf of the NN false measurement z_F. Clearly, z_F concentrates much more around the gate center than a uniform random variable does.

C. Probabilities of Data Association Events

The (prior) probabilities of the events of no validated measurement (M_0), of correct data association (M_T), and of incorrect association (M_F) are needed in the next section as weightings in calculation of the average estimation errors. The last two probabilities also serve as normalization constants in the conditional pdfs of the NN measurement obtained above. These probabilities are also useful for other purposes, such as to develop improved versions of the NN filter.

THEOREM 3 *With Assumptions A1–A6, the probabilities of the three mutually exclusive and collectively exhaustive events conditioned on Z^{k-1} (omitted below) are given by, respectively,*

$$P\{M_0(k)\} = (1 - P_D P_G) e^{-\lambda V_\gamma(k)} \tag{36}$$

$$P\{M_T(k)\} = \frac{P_D}{2^{n/2}\Gamma(n/2)} \int_0^\gamma D^{n/2-1} \times e^{-\lambda V_D(k) - D/2} dD \le P_D P_G \tag{37}$$

$$\begin{aligned} P\{M_F(k)\} &= \frac{n\lambda}{2} \int_0^\gamma \frac{V_D(k) e^{-\lambda V_D(k)}}{D} \times [1 - P_D P\{\chi^2(n) \le D\}] dD \\ &= 1 - (1 - P_D P_G) e^{-\lambda V_\gamma} - P\{M_T(k)\} \end{aligned} \tag{38}$$

where

$$P_G = P\{\chi^2(n) \le \gamma\} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\gamma D^{n/2-1} e^{-D/2} dD \le P_D P_G. \tag{39}$$

PROOF See Appendix.

REMARKS 1) These three event probabilities do sum up to unity (see Appendix). 2) These event probabilities are time varying only because they are functions of $|S(k)|^{1/2}$, which can be determined off-line if an off-line recursion is available for $P(k \mid k)$. In such a case, these probabilities can be obtained off-line. 3) In a clean (clutterless) environment (i.e., $\lambda = 0$), these probabilities become, as they should be,

$$\lim_{\lambda \to 0} P\{M_0(k)\} = 1 - P_D P_G$$

$$\lim_{\lambda \to 0} P\{M_T(k)\} = P_D P_G \tag{40}$$

$$\lim_{\lambda * 0} P\{M_F(k)\} = 0.$$

4) In general the evaluation of the above integrals requires the use of numerical methods. In the common case of two-dimensional measurements ($n = 2$), however, analytical expressions can be readily obtained as

$$P\{M_T(k)\} = \frac{P_D}{2\alpha}(1 - e^{-\alpha\gamma}) \tag{41}$$

$$P\{M_F(k)\} = (1 - P_D)(1 - e^{-\beta\gamma}) + \frac{P_D\beta}{\alpha}(1 - e^{-\alpha\gamma}) \tag{42}$$

where

$$\beta = \lambda c_n |S(k)|^{1/2} \qquad \alpha = \beta + \tfrac{1}{2} \tag{43}$$

and use has been made, for $n = 2$, of

$$P_G = 1 - e^{-\gamma/2}. \tag{44}$$

Note that (41) coincides with the expression given in [18], where a rectangular gate is used, when the gate size is infinity. 5) It is interesting to compare the event probabilities given here with those obtained in [14] using a particular kind of "optimal" assignment algorithm when the gate size goes to infinity. Three closed forms of the integral involved in the event probabilities are given in [14] for $n = 1$, 2, and 4, respectively. 6) After this paper has been accepted, we noticed that an equivalent form of (37) was given in [4] without proof.

IV. PROPAGATION OF ESTIMATION ERROR AND OFF-LINE RECURSIVE PERFORMANCE PREDICTION

In this section, we first derive the propagation of the *predicted matrix mean square error* (MSE) [2] of state estimation for the NN filter. Then an off-line performance predictor is presented. This predictor is based on the current-mode-conditional (CMC) version of the HYCA technique, presented in [9–11, 13], for hybrid algorithms—those with both continuous-valued and discrete-valued uncertainties. The NN filter falls into this category because the performance of the algorithm depends not only on continuous-valued uncertainties (i.e., the process noise and the measurement noise) but also on a discrete-valued uncertainty—the three random events concerning the origin of the NN measurement.

In order to apply the HYCA technique, it is essential to identify the *algorithm-assumed mode set* and the *system mode set*[5] first. It is not difficult to see that the NN filter can be viewed as actually having three modes corresponding to the three events M_0, M_T, and M_F, respectively, since the performance of the filter is significantly different under each of these three events, even though the filter itself assumes only two modes corresponding to M_0 and M_T, respectively. Here we assume that the true system dynamics and the NN filter assumed dynamics are identical. For convenience, let $M_1(k)$ and $M_2(k)$ denote $M_T(k)$ and $M_F(k)$, respectively.

The above-mentioned recursion is based on the following decomposition of the predicted matrix square error or MSE:[6]

$$\begin{aligned} E[\mathrm{MSE}(k) \mid Z^{k-1}] &= E[\mathrm{MaSE}(k) \mid Z^{k-1}] \\ &= \sum_{i=0}^{2} E[\mathrm{MaSE}(k) \mid M_i(k), Z^{k-1}] P\{M_i(k) \mid Z^{k-1}\} \end{aligned} \tag{45}$$

where the matrix square error is defined as

$$\mathrm{MaSE}(k) \stackrel{\Delta}{=} [x(k) - \hat{x}(k \mid k)][x(k) - \hat{x}(k \mid k)]' \tag{46}$$

and the matrix MSE is defined[7] as

$$\mathrm{MSE}(k) \stackrel{\Delta}{=} E\{[x(k) - \hat{x}(k \mid k)][x(k) - \hat{x}(k \mid k)]' \mid Z^k\}. \tag{47}$$

Note that the expectation of MaSE is equal to that of MSE, both conditioned on Z^{k-1}. We use the term *predicted matrix* MSE for this expectation because 1) what we want to predict is the *mean* or *average* error rather than the random actual error and 2) the term expected (or predicted) MaSE is ambiguous since it can imply conditioning on either Z^k or Z^{k-1}. With this convention, in what follows the conditioning on Z^{k-1} is dropped for simplicity.

THEOREM 4 *With Assumptions A1–A6, the predicted matrix MSE when the NN measurement is not from the target of interest is*

$$E[\mathrm{MaSE}(k) \mid M_F(k)] = P(k \mid k-1, M_F) + \frac{c_F(k)}{P\{M_F(k)\}} \times W(k)S(k)W(k)' \tag{48}$$

where $S(k)$ *and* $W(k)$ *were defined by* (16) *and* (17)*, respectively;*

$$P(k \mid k-1, M_F) = E\{[x(k) - \hat{x}(k \mid k-1)] \cdot [x(k) - \hat{x}(k \mid k-1)]' \mid M_F(k)\} \tag{49}$$

[5]The system mode set is the set of possible system behavior patterns while the algorithm-assumed mode set is the set of possible system behavior patterns assumed either explicitly or implicitly in the algorithm.

[6]This decomposition of the predicted matrix MSE into individual ones conditioned on each of the possible system modes is the key idea of the CMC version of the HYCA technique.

[7]Note that this is not the covariance since the estimate $\hat{x}$ is in general not the conditional mean.

and $c_F(k)$ is a scalar, given by

$$c_F(k) = \frac{\beta}{2}\int_0^\gamma q^{n/2} e^{-\beta q^{n/2}}[1 - P_D P\{\chi^2(n) \le q\}]\,dq \tag{50}$$

where β is the expected number of false measurements in a 1-σ gate, given before.

PROOF See Appendix.

REMARKS 1) Since $c_F(k) \ge 0$, the second term in the right-hand side (RHS) of (48), which is at least positive semi-definite, quantifies the predicted average increase in estimation error as a result of using a false measurement to update the estimate. It should not be surprising that this term does not vanish[8] in a clean environment since the predicted matrix MSE of (48) is *conditioned on $M_F(k)$*. In fact, the corresponding term in the unconditional (with respect to (wrt) M_F) predicted matrix MSE of (45), which is the above-mentioned second term times $P\{M_F(k)\}$, does vanish when $\lambda = 0$, as it should be. 2) As shown later, it turns out that the exact value of $P(k \mid k-1, M_F)$ is immaterial for this study.

COROLLARY 4.1 *In the common case of two-dimensional measurements ($n = 2$), the scalar $c_F(k)$ is given by*

$$c_F(k) = \begin{cases} \dfrac{\beta}{2}\left\{\dfrac{P_D}{\alpha^2}[1-(1+\alpha\gamma)e^{-\alpha\gamma}] + \dfrac{1-P_D}{\beta^2}[1-(1+\beta\gamma)e^{-\beta\gamma}]\right\} & \lambda > 0 \\ 0 & \lambda = 0 \end{cases} \tag{51}$$

where α and β were defined in (43).

PROOF This follows directly from Theorem 4 by setting $n = 2$.

Note that $C_F(k)$ is actually continuous in λ even though a piecewise formula is given, which is useful for computer computation.

THEOREM 5 *With Assumptions A1–A6, the predicted matrix MSE when the NN measurement is from the target of interest is*

$$E\{\text{MaSE}(k) \mid M_T(k)\} = P(k \mid k-1) - W(k)S(k)W(k)'$$

$$= P(k \mid k-1, M_T) \tag{52}$$

$$- \frac{c_T(k)}{P\{M_T(k)\}} W(k)S(k)W(k)' \tag{53}$$

where $c_T(k) \le P\{M_T(k)\}$ is a scalar, given by

$$c_T(k) = \frac{P_D}{2^{n/2+1}\Gamma(n/2+1)}\int_0^\gamma q^{n/2} e^{-\beta q^{n/2} - q/2}\,dq. \tag{54}$$

PROOF See Appendix.

REMARKS 1) In the limit as $\lambda \to 0$ and $\gamma \to \infty$, $c_T(k) = P\{M_T(k)\} = P_D$ and $P(k \mid k-1, M_T) = P(k \mid k-1)$, as they should be. 2) As shown later, it turns out that the exact value of $P(k \mid k-1, M_T)$ is immaterial for this study.

COROLLARY 5.1 *In the common case of $n = 2$, the scalar $c_T(k)$ of* (54) *is given by*

$$c_T(k) = \frac{P_D}{4\alpha^2}[1-(1+\alpha\gamma)e^{-\alpha\gamma}]. \tag{55}$$

PROOF See Appendix.

Finally, combining the results presented in this section so far, we have the following important result.

THEOREM 6 *With the assumptions of Theorem* 5, *the predicted matrix MSE defined in* (45) *is*

$$\begin{aligned} E[\text{MSE}(k)] = P(k \mid k-1) \\ -[+c_T(k) - c_F(k)] \\ \times W(k)S(k)W(k)' \end{aligned} \tag{56}$$

where $c_F(k)$ and $c_T(k)$ are the scalars defined in Theorems 4 *and* 5, *respectively.*

PROOF See Appendix.

REMARKS 1) Equation (56) reveals quantitatively the relationship between the filter performance and certain scenario parameters, such as the false alarm density, the target detection probability, and the gate size. 2) Since the NN filter will never outperform the optimal Kalman filter that uses always the correct measurements, we have the following inequality:

$$u = +c_T(k) - c_F(k) < 1. \tag{57}$$

In a manner similar to the investigation of the probabilistic data association filter [3, 13], u can be called the *information reduction factor* since it reflects the increase in the uncertainty of the estimate to account for the uncertainty in the measurement origin.

COROLLARY 6.1 *For $n = 2$, as the gate size increases the information reduction factor tends to its limit, given by*

$$\lim_{\gamma\to\infty} u = \begin{cases} P_D\dfrac{1-2\beta}{4\alpha^2} - \dfrac{1-P_D}{2\beta} & \beta \ne 0 \\ P_D & \beta = 0 \end{cases}. \tag{58}$$

PROOF This follows directly from Theorem 6, Corollary 4.1 and (41). □

[8]The λ factor outside the exponential in $c_F(k)$ and in $P\{M_F(k)\}$ cancels out.

$$
\begin{aligned}
\overline{P}(k|k-1) &= F(k-1)\overline{P}(k-1|k-1)F(k-1)' + G(k-1)Q(k-1)G(k-1)' \\
\overline{S}(k) &= H(k)\overline{P}(k|k-1)H(k)' + R(k) \\
\overline{W}(k) &= \overline{P}(k|k-1)H(k)'\overline{S}(k)^{-1} \\
\overline{\beta} &= \lambda c_n |\overline{S}(k)|^{1/2} \\
\overline{P}\{M_T(k)\} &= \int_0^{\gamma} \frac{P_D D^{n/2-1}}{2^{n/2}\Gamma(n/2)} \exp\left[-\overline{\beta} D^{n/2} - \frac{1}{2}D\right] dD \\
\overline{c}_F(k) &= \frac{\overline{\beta}}{2}\int_0^{\gamma} q^{n/2} e^{-\overline{\beta} q^{n/2}} \left[1 - P_D P\{\chi^2(n) \le q\}\right] dq \\
\overline{c}_T(k) &= \frac{P_D}{2^{n/2+1}\Gamma(n/2+1)} \int_0^{\gamma} q^{n/2} e^{-\overline{\beta} q^{n/2} - q/2} dq \\
\overline{P}(k|k) &= \overline{P}(k|k-1) - [\overline{c}_T(k) - \overline{c}_F(k)]\overline{W}(k)\overline{S}(k)\overline{W}(k)' \\
\overline{P}_{\mathrm{NNF}}(k|k) &= \overline{P}(k|k-1) + \left[1 - (1 - P_D P_G)e^{-\overline{\beta}\gamma^{n/2}}\right]\overline{W}(k)\overline{S}(k)\overline{W}(k)'
\end{aligned}
$$

Fig. 2. Off-line recusion for predicted matrix MSE and filter-calculated covariance in NN filter.

THEOREM 7 *The expectation of the filter-calculated covariance matrix is given by*

$$
\begin{aligned}
E[P(k \mid k) \mid Z^{k-1}] &= P(k \mid k-1) - [1 - (1 - P_D P_G)e^{-\lambda V_\gamma(k)}] \\
&\quad \times W(k)S(k)W(k)'. \qquad (59)
\end{aligned}
$$

PROOF It is clear from (14) and (19) that

$$
\begin{aligned}
E[P(k \mid k) \mid Z^{k-1}] &= P(k \mid k-1) - [1 - P\{M_0(k)\}] \\
&\quad \times W(k)S(k)W(k)'. \qquad (60)
\end{aligned}
$$

Hence, Theorem 7 follows from (36). □

REMARK This expectation predicts what the NN filter *believes its performance is*[12] whereas the expectation given in (56) predicts what the performance of the NN filter *really is*. They are usually very different, especially in the transient of tracking divergence.

The results presented so far are exact under the corresponding assumptions. While these assumptions are generally considered precise in the first scan, some of them (e.g., A3) do not hold exactly in later scans because the NN filter does not yield the conditional mean of the state. As a result, a practical multiscan performance prediction involves some approximation. It should be kept in mind that the performance predictor to be presented below assumes that the above single-scan analysis can be extended to a multiscan performance prediction. Specifically, it assumes that all the assumptions are valid for later scans as well as for the first one and thus the expected matrix MSE in the previous scan k is given exactly by (56).

Off-Line Recursive Performance Predictor: Based on the above theorems for the NN filter, an off-line recursion for the predicted matrix MSE of the state estimate and its covariance calculated by the filter is given in Fig. 2. An overbar notation is used to denote the off-line quantities and $\overline{P}_{\mathrm{NNF}}(k \mid k)$ is the off-line counterpart of the predicted covariance defined in (59). Also, the effects of the false measurement density and the target detection probability on the tracking error become clear from this recursion.

Note that the predictor of Fig. 2 is based on a simplified version of the CMC HYCA technique. A more accurate predictor can be obtained by applying the nonsimplified CMC HYCA technique [10, 13]. The difference between the simplified and non-simplified CMC HYCA is analogous to that between the generalized pseudo-Bayesian algorithm of first order (GPB1) and the interacting multiple model (IMM) algorithm. An even more accurate predictor can be obtained based on the more sophisticated mode-sequence-conditional HYCA technique [10, 11], which requires, however, the specification of a sequence of true mode sequence, that is, a sequence of the occurrences of either M_T, M_F, or M_0.

V. A NUMERICAL EXAMPLE

Consider a nearly constant velocity model with position-only measurements, described by (8) and (9) with

$$
F(k) = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (61)
$$

$$
G(k) = \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ T & 0 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \end{bmatrix} \qquad (62)
$$

$$
H(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (63)
$$

$$
\overline{v}(k) = 0 \qquad Q(k) = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \qquad (64)
$$

$$
R(k) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \qquad (65)
$$

where T is the sampling interval; x is the state of the target, defined as

$$
x = [\xi \quad \dot{\xi} \quad \eta \quad \dot{\eta}]' \qquad (66)
$$

with ξ and η denoting the orthogonal coordinates of the horizontal plane. This model can be normalized by choosing T as the unit of time and $\sqrt{r}$ as the unit of distance to yield a model of target dynamics and measurement [19], again described by (8) and (9), but

[12]In other words, it predicts how "honest" the filter is in the particular case.

with

$$F(k) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{67}$$

$$G(k) = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \tag{68}$$

$$H(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{69}$$

$$\bar{v}(k) = 0 \qquad Q(k) = \begin{bmatrix} q^* & 0 \\ 0 & q^* \end{bmatrix} \tag{70}$$

$$R(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{71}$$

and a dimensionless state

$$x = \begin{bmatrix} \frac{\xi}{\sqrt{r}} & \frac{\dot{\xi}T}{\sqrt{r}} & \frac{\eta}{\sqrt{r}} & \frac{\dot{\eta}T}{\sqrt{r}} \end{bmatrix}' \tag{72}$$

where $q^* = qT^4/r$ is equal to the squared value of the target maneuvering index [2, 3].

This *canonical* model [19] was used in our numerical examples for the generation of the true state and measurements. The NN filter and our performance predictor were coded in MATLAB. A Poisson number generator was implemented, tested, and then used for the generation of the number of false measurements. The efficient algorithm presented in [7] was used to generate false measurements uniformly distributed inside the elliptic gate.

The following scenario was used to obtain the results illustrated below: $q^* = 0.16$, $\lambda^* = 0.05$, $\gamma = 16$, $P_D = 0.7$, and the target started from the origin with a speed of 10 at an angle which is uniformly distributed over $[0, 2\pi)$. Note that $\lambda^* = 0.05$ for this canonical model is equivalent to having a $\lambda = \lambda^*/r$ in a situation where the measurement standard deviation is $\sqrt{r}$. All simulation results presented here are based on 500 Monte Carlo runs.

Fig. 3 shows the actual (coordinate-combined) position rms estimation error, the filter calculated on-line position error standard deviation, the predictor predicted position error, and the predictor predicted position error standard deviation. Note that the vertical unit is $\sqrt{r}$, i.e., the measurement standard deviation. The corresponding comparison for velocity estimation errors is shown in Fig. 4. The vertical unit for this figure is $\sqrt{r}/T$. Fig. 5 shows the comparison between the actual frequencies of correct and incorrect data associations and their corresponding probabilities.

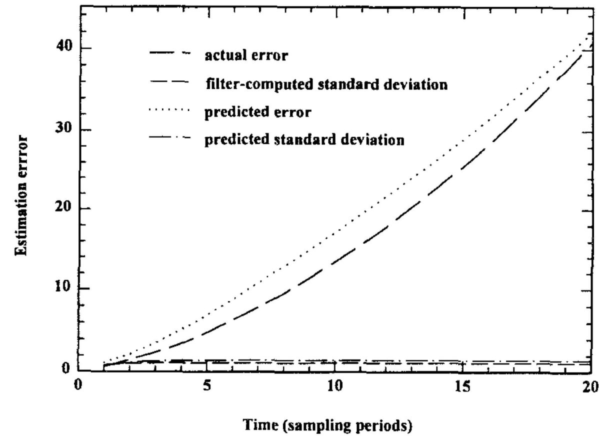

Fig. 3. Comparison of position rms errors.

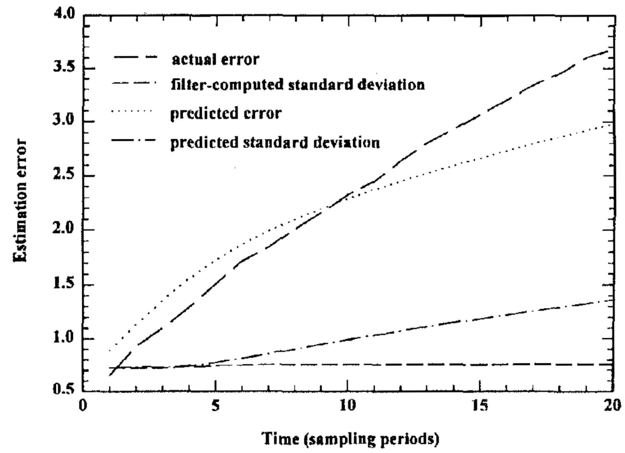

Fig. 4. Comparison of velocity rms errors.

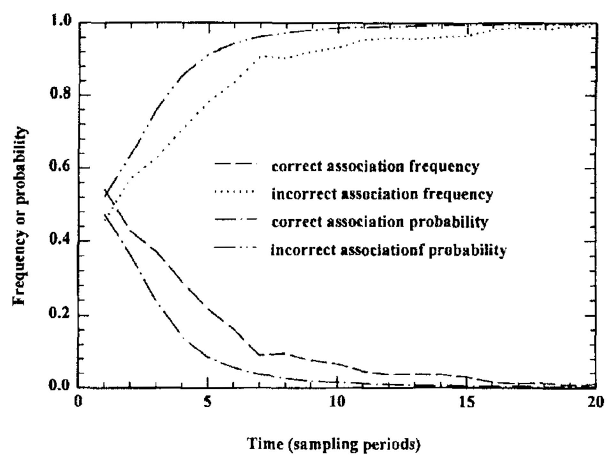

Fig. 5. Comparison of frequencies and probabilities.

It is clear from the figures that both the actual performance of the filter and the filter-believed performance were predicted faithfully by our performance predictor that did not use even a single measurement. Note that the difference between the actual performance and its prediction tends to increase after the estimation error becomes very large. This is due to the fact that in this case the track has already been lost and thus some of our assumptions (e.g., A3) are no longer valid. There is not much value, however, to predict the behavior of the NN filter after it has already lost the target.

VI. SUMMARY

The NN filter is the most common algorithm currently in use for tracking in clutter. It is either used alone or as a module in a more complex algorithm. Efforts have been made to analyze its performance (e.g., [16, 17]). It is, however, difficult to quantify the dynamic process of the important tracking divergence phenomenon for this filter. A technique that can provide such a quantification without the need of stochastic simulations has been developed in this paper. This technique gives consideration to the important scenario dependence of the performance. An off-line recursion of the tracking error has been obtained. This has been verified via a comparison with results from Monte Carlo simulations. In addition to the capability of faithfully predicting the filter performance, the recursion equations reveal quantitatively the dependence of the filter performance on the false alarm rate and the target detection probability. This recursion is based on a general approach to performance prediction of algorithms with continuous and discrete uncertainties developed recently [9, 10]. Also presented here are the conditional pdfs of the NN measurement under the events of a correct and incorrect data association, respectively, the probabilities of these data association events, and the propagation of the matrix mean square error conditioned on these events. The development of the above-mentioned recursion relies heavily on these conditional pdfs and probabilities.

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APPENDIX

In this Appendix, unless otherwise required for clarity, 1) the conditioning on Z^{k-1} is dropped; 2) the time argument k is omitted; 3) the region for the measurement y and NDS D are R_γ and $[0,\gamma]$, respectively, and are omitted. The pdfs and probabilities outside these regions are all zero and are not stated redundantly.

A. Proof of Theorem 1

Denote the joint probability pdf of an event X and a random variable z by [2]

$$P[X,y] = \lim_{\Delta y\to 0} \frac{P\{X, y < z \le y+\Delta y\}}{\Delta y}. \tag{73}$$

Define also two events

$$A = \{\text{Target is detected}\} \tag{74}$$

$$\begin{aligned} B &= \{\text{The target-originated measurement is validated}\} \\ &= \{z_T \in R_\gamma\}. \end{aligned} \tag{75}$$

Thus, we have

$$P\{A\} = P_D \qquad P\{B \mid A\} = P_G. \tag{76}$$

With Assumptions A1–A4, the pdf of z_*, conditioned on Z^{k-1} (omitted) and M_T, is given by

$$\begin{aligned} p_{z_*}(y \mid M_T) &= p_{z_T}(y \mid M_T, A, B) \\ &= \frac{P[M_T, y \mid A, B]}{P\{M_T \mid A, B\}} \\ &= \frac{P\{M_T \mid y, A, B\} p(y \mid A, B)}{P\{M_T \mid A, B\}} \\ &= \frac{P\{\text{no false measurement in } R_{D(y)}\} P[y, A, B]/P\{A, B\}}{P\{M_T\}/P\{A,B\}} \\ &= \frac{\mu_F(0, V_{D(y)}) P[y, A, z_T \in R_\gamma]}{P\{M_T\}} \\ &= \frac{e^{-\lambda V_{D(y)}} P\{A\} P[y, z_T \in R_\gamma \mid A]}{P\{M_T\}} \\ &= \frac{e^{-\lambda V_{D(y)}} P_D \mathcal{N}[y; \hat{z}(k \mid k-1), S] U(y; R_\gamma)}{P\{M_T\}} \end{aligned} \tag{77}$$

where z_T is the validated measurement from the target; R_D is the ellipsoid defined in (6); and

$$\mu_F(m, V) = \frac{(\lambda V)^m}{m!} e^{-\lambda V} \tag{78}$$

denotes the Poisson pmf that there are m false measurements with the spatial density λ in a region of volume V. □

B. Proof of Proposition 1

From (7), one has

$$dV_D = \frac{nV_D}{2D} dD. \tag{79}$$

In view of (28), one has

$$p_D(D) = K p_z[D(z)] = K p_z(z) \tag{80}$$

where $K = nV_D/2D$ is a normalization constant, which can be obtained by the following normalization property of the pdfs:

$$\begin{aligned} \int_{R_\gamma} p_D[D(z)] \frac{2D}{nV_D} dz &= \int_0^\gamma p_D(D) dD \\ &= 1 = \int_{R_\gamma} p_z(z) dz \end{aligned} \tag{81}$$

where the first equality in the above equation follows from (79). □

C. Proof of Corollary 1.1

It is straightforward to verify that (31) and (30) are equivalent. Since $p_{z_*}(y)$ is elliptically symmetric, (31) follows directly from Theorem 1 and Proposition 1. Alternatively, Corollary 1.1 can be proven directly as follows.

With Assumptions A1–A4, one has, denoting $D_T = D(Z_T)$,

$$p_{D_*}[D \mid M_T] = \frac{p_{D_T}(D)}{P\{M_T\}} P\{M_T \mid D_T = D\}$$

$$= \frac{P_D\, p_{\chi^2(n)}(D)}{P\{M_T\}}$$

$$\times P\{\text{no false measurement in ellipsoid} R_D\}$$

$$= \frac{P_D}{P\{M_T\}} \frac{D^{n/2-1}e^{-D/2}}{2^{n/2}\Gamma(n/2)} \mu_F(0, V_D)$$

$$= \frac{P_D}{P\{M_T\}} \frac{D^{n/2-1}e^{-D/2}}{2^{n/2}\Gamma(n/2)} e^{-\lambda V_D} \tag{82}$$

where $p_{D_T}(D)$ is the pdf of $D(z_T)$—a chi-square pdf. $\square$

D. Proof of Proposition 2

This proposition follows from (88) and Proposition 1 directly. However, we also provide the following more constructive proof.

Let z_j be an arbitrary member of Z_F, the set of false measurement in the gate at a particular time. Then one has, with A4–A5,

$$p_{z_F}(y) = \lim_{\Delta y \to 0} \frac{P\{y < z_j \le y + \Delta y, D(z_j) = \min_z D(z),\ \text{for some } j \mid z_F \in R_\gamma\}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \sum_{m=1}^{\infty} P \left\{ \begin{matrix} y < z_j \le y + \Delta y, \\ D(z_j) = \min_{z \in Z_F} D(z) \le \gamma, \end{matrix} \quad \text{for some} \quad j \mid m_F = m \right\} \frac{P\{m_F = m\}}{P\{D_F \le \gamma\}}$$

$$= \sum_{m=1}^{\infty} \binom{m}{1} P \left\{ D(z_j) = \min_{z \in Z_F} D(z) \le \gamma \mid z_j = y,\ m_F = m \right\} P\{m_F = m\} \frac{p_{z_j}(y \mid m_F = m)}{1 - e^{-\lambda V_\gamma}}$$

$$= V_\gamma^{-1} \sum_{m=1}^{\infty} \frac{m[1 - P\{D_i \le D\}]^{m-1}}{1 - e^{-\lambda V_\gamma}} P\{m_F = m\}$$

$$= V_\gamma^{-1} \frac{p_{D(z_F)}(D(y))}{p_{D_j}(D(y))}$$

$$= \frac{\lambda e^{-\lambda V_{D(y)}}}{1 - e^{-\lambda V_\gamma}} \tag{83}$$

in which use has been made of (90)–(93) and the following

$$p_{z_j}(y \mid m_F = m) = p_{z_j}(y) = V_\gamma^{-1}. \tag{84}$$

$\square$

E. Proof of Lemma 1

Let D_F be the smallest NDS of a false validated measurement, that is, the NDS corresponding to the false validated measurement which is closest to the gate center. Note first that

$$P\{D_F \le \gamma\} = P\{\text{at least one false measurement in } R_\gamma\}$$

$$= 1 - \mu_F(0, V_\gamma)$$

$$= 1 - e^{-\lambda V_\gamma}. \tag{85}$$

For notational simplicity, the conditioning on $0 \le D_F \le \gamma$ is dropped in what follows whenever D is involved. For example,

$$P\{D_F \le D\} \stackrel{\Delta}{=} P\{D_F \le D \mid 0 \le D_F \le \gamma\}$$

$$= \frac{1}{c} P\{\text{at least one false measurement in } R_D\}$$

$$= \frac{1}{c}[1 - \mu_F(0, V_D)]$$

$$= \frac{1 - e^{-\lambda V_D}}{1 - e^{-\lambda V_\gamma}}. \tag{86}$$

We are now ready to prove Lemma 1. Since

$$p_{D_*}[D \mid M_F] = p_{D_*}[D \mid M_F, D_F \le \gamma]$$

$$= \frac{P\{M_F \mid D_F = D\} p_{D_F}(D)}{P\{M_F \mid D_F \le \gamma\}}$$

$$= \frac{p_{D_F}(D)(1-e^{-\lambda V_\gamma})}{P\{M_F\}}$$

$$\times P\{z_T \text{ is not detected or is outside } R_D \mid D_F = D\}$$

$$= \frac{p_{D_F}(D)(1-e^{-\lambda V_\gamma})}{P\{M_F\}}[1 - P_D P\{\chi^2(n) \le D\}] \tag{87}$$

where $p_{D_F}(D)$ is the pdf of D_F, comparing with (33) it suffices to prove the following:

$$p_{D_F}(D) = \frac{n\lambda V_D}{2D(1-e^{-\lambda V_\gamma})} e^{-\lambda V_D}. \tag{88}$$

Clearly, (88) follows from taking derivative of (86):

$$p_{D_F}(D) = \frac{d}{dD} P\{D_F \le D\} = \frac{n\lambda V_D}{2D(1-e^{-\lambda V_\gamma})} e^{-\lambda V_D}. \tag{89}$$

A more constructive proof of (88) is the following, which serves to demonstrate some important concepts and results associated with the NN filters.

Let D_j be the NDS of an arbitrary false measurement in the gate. It has been shown in [1] that its cdf and pdf are, respectively,

$$P\{D_j \le D\} = \frac{V_D}{V_\gamma} = \left(\frac{D}{\gamma}\right)^{n/2} \tag{90}$$

$$p_{D_j}(D) = \frac{n}{2D}\left(\frac{D}{\gamma}\right)^{n/2}. \tag{91}$$

Denote by m_F the number of false measurements in the gate at a particular time. It can be shown from order statistics [15] that the pdf of D_F is

$$p_{D_F}(D \mid m_F = m) = \frac{m[1 - P\{D_j \le D\}]^{m-1} p_{D_j}(D)}{1 - e^{-\lambda V_\gamma}} \qquad 0 \le D \le \gamma \tag{92}$$

where the denominator is the normalization constant accounting for the fact that $p_{D_F}(D \mid m_F = m) = 0$ if $D_F \notin [0,\gamma]$. Hence, one has

$$p_{D_F}(D) = \sum_{m=1}^{\infty} p_{D_F}(D \mid m_F = m) P\{m_F = m\}$$

$$= \sum_{m=1}^{\infty} m\left[1 - \left(\frac{D}{\gamma}\right)^{n/2}\right]^{m-1} \frac{n(D/\gamma)^{n/2}}{2D(1-e^{-\lambda V_\gamma})} \frac{(\lambda V_\gamma)^m}{m!} e^{-\lambda V_\gamma}$$

$$= \frac{e^{-\lambda V_\gamma}}{1-e^{-\lambda V_\gamma}} \frac{d}{dD} \sum_{m=1}^{\infty}$$

$$\times \left\{ \int m\left[1 - \left(\frac{D}{\gamma}\right)^{n/2}\right]^{m-1} \frac{n}{2D}\left(\frac{D}{\gamma}\right)^{n/2} dD \right\} \frac{(\lambda V_\gamma)^m}{m!}$$

$$= -\frac{e^{-\lambda V_\gamma}}{1-e^{-\lambda V_\gamma}} \frac{d}{dD} e^{\lambda V_\gamma[1-(D/\gamma)^{n/2}]}$$

$$= \frac{n\lambda V_D}{2D(1-e^{-\lambda V_\gamma})} e^{-\lambda V_D}. \tag{93}$$

□

F. Proof of Theorem 2

This theorem follows from Proposition 1 and Lemma 1.

G. Proof of Theorem 3

Clearly, (36) follows from Assumptions A1, A2, and A4 as follows:

$$P\{M_0\}$$

$$= P\{\text{no target-originated and false validated measurement}\}$$

$$= P\{\text{no target-originated validated measurement}\}$$

$$\times P\{\text{no false validated measurement}\}$$

$$= (1 - P_D P_G) P\{m_F = 0\}. \tag{94}$$

Equation (37) follows from the normalization property of the pdf of (30), that is, the integral of $p_{D_*}(D \mid M_T)$ over all D must be equal to unity. The inequality in (37) follows from the fact that

$$e^{-\lambda V_D} \le 1. \tag{95}$$

The first equation in (38) follows from the normalization property of the pdf of (33). The second equation follows from the following:

$$\int_0^\gamma \frac{n\lambda V_D e^{-\lambda V_D}}{2D}[1 - P_D P\{\chi^2(n) \le D\}] dD$$

$$= \int_0^\gamma \frac{n\lambda V_D e^{-\lambda V_D}}{2D} dD$$

$$+ P_D \int_0^\gamma P\{\chi^2(n) \le D\} d\left(e^{-\lambda V_D}\right)$$

$$= -e^{-\lambda V_D}\Big|_0^\gamma + P_D e^{-\lambda V_D} P\{\chi^2(n) \le D\}\Big|_0^\gamma$$

$$- P_D \int_0^\gamma e^{-\lambda V_D} \frac{D^{n/2-1} e^{-\frac{1}{2}D}}{2^{n/2}\Gamma(n/2)} dD$$

$$= 1 - e^{-\lambda V_\gamma} + P_D P_G e^{-\lambda V_\gamma} - P\{M_T\} \tag{96}$$

where use has been made of (37) and the following:

$$P\{\chi^2(n) \leq \gamma\} = P_G. \tag{97}$$

Note that the probabilities of the three events do sum up to unity. □

H. Proof of Theorem 4

Under M_F one has

$$\begin{aligned} E[\mathrm{MSE} \mid M_F] &= P(k \mid k-1, M_F) + WE[\nu\nu' \mid M_F]W' \\ &\quad - E\{[x - \hat{x}(k \mid k-1)]\nu' \mid M_F\}W' \\ &\quad - WE\{\nu[x - \hat{x}(k \mid k-1)]' \mid M_F\} \\ &= P(k \mid k-1, M_F) + WE[\nu\nu' \mid M_F]W' \end{aligned} \tag{98}$$

where the last equation above follows from the Assumptions A5–A6, which implies that ν is independent of the state prediction error and uniformly distributed in the gate (and thus has a zero mean).

Now let us establish the following lemma first.

LEMMA 2 *The covariance of the measurement residual under M_F is*

$$E[\nu\nu' \mid M_F] = \frac{c_F}{P\{M_F\}} S \tag{99}$$

where c_F is a scalar, given by (50)*, and S was defined by* (16).

PROOF Since S is positive definite, let

$$\tilde{\nu} = S^{-1/2}\nu \tag{100}$$

such that

$$D(\nu) = \nu' S^{-1} \nu = \tilde{\nu}'\tilde{\nu} = \|\tilde{\nu}\|^2 \tag{101}$$

and by Theorem 2

$$\begin{aligned} p(\tilde{\nu} \mid M_F) &= |S|^{1/2} \frac{\lambda e^{-\lambda V_{\|\tilde{\nu}\|^2}}}{P\{M_F\}} \\ &\quad \times [1 - P_D P\{\chi^2(n) \leq \|\tilde{\nu}\|^2\}] U(\tilde{\nu}; \tilde{R}_\gamma) \end{aligned} \tag{102}$$

where

$$\tilde{R}_\gamma = \{\tilde{\nu} : \|\tilde{\nu}\|^2 \leq \gamma\}. \tag{103}$$

Hence,[13]

$$E[\nu\nu' \mid M_F] = S^{1/2} E[\tilde{\nu}\tilde{\nu}' \mid M_F] S^{1/2}. \tag{104}$$

[13]Every positive definite symmetric matrix has a symmetric square root matrix.

Clearly, one has

$$\begin{aligned} E[\tilde{\nu}\tilde{\nu}' \mid M_F] &= \int_{\tilde{R}_\gamma} \tilde{\nu}\tilde{\nu}' p(\tilde{\nu} \mid M_F) d\tilde{\nu} \\ &= \frac{\lambda \mid S \mid^{1/2}}{P\{M_F\}} \int_{\tilde{R}_\gamma} \tilde{\nu}\tilde{\nu}' e^{-\lambda V_{\|\tilde{\nu}\|^2}} \\ &\quad \times [1 - P_D P\{\chi^2(n) \leq \|\tilde{\nu}\|^2\}] d\tilde{\nu}. \end{aligned} \tag{105}$$

Since $p(\tilde{\nu} \mid M_F)$ is even in the components of $\tilde{\nu}$ (i.e., $\tilde{\nu}_i$, $i = 1,\ldots,n$) and the off-diagonal elements are odd in $\tilde{\nu}_i$, $i = 1,\ldots,n$, the above integral (covariance) is a diagonal matrix. In addition, because of the symmetry of $p(\tilde{\nu} \mid M_F)$ in $\tilde{\nu}_i$, $i = 1,\ldots,n$, one has

$$E[\tilde{\nu}\tilde{\nu}' \mid M_F] P\{M_F\} = c_F I \tag{106}$$

where I is the identity matrix.

Since

$$\begin{aligned} E[\tilde{\nu}'\tilde{\nu} \mid M_F] &= E[\mathrm{tr}(\tilde{\nu}'\tilde{\nu}) \mid M_F] = E[\mathrm{tr}(\tilde{\nu}\tilde{\nu}') \mid M_F] \\ &= \mathrm{tr}(E[\tilde{\nu}\tilde{\nu}' \mid M_F]) = \frac{nc_F}{P\{M_F\}} \end{aligned} \tag{107}$$

one has

$$c_F = \frac{P\{M_F\}}{n} E[\tilde{\nu}'\tilde{\nu} \mid M_F]. \tag{108}$$

Thus, Lemma 2 follows from the following

$$\begin{aligned} E[\tilde{\nu}'\tilde{\nu} \mid M_F] P\{M_F\} &= \lambda |S|^{1/2} \int_{\tilde{R}_\gamma} \tilde{\nu}\tilde{\nu}' e^{-\lambda V_{\|\tilde{\nu}\|^2}} \\ &\quad \times [1 - P_D P\{\chi^2(n) \leq \|\tilde{\nu}\|^2\}] d\tilde{\nu} \\ &= \frac{n\lambda c_n |S|^{1/2}}{2} \int_0^\gamma q^{n/2} e^{-\lambda c_n |S|^{1/2} q^{n/2}} \\ &\quad \times [1 - P_D P\{\chi^2(n) \leq q\}] dq. \end{aligned} \tag{109}$$

□

Finally, Theorem 4 follows directly from this lemma and (98). □

I. Proof of Theorem 5

Let

$$\begin{aligned} \tilde{x} &= x(k) - \hat{x}(k \mid k) \\ \overline{x} &= x(k) - \hat{x}(k \mid k-1) \\ \tilde{P} &= P(k \mid k-1) - W(k)S(k)W(k)'. \end{aligned}$$

Then, one has

$$\begin{aligned}
E[\mathrm{MaSE} \mid M_T] &= E[\tilde{x}\tilde{x}' \mid Z^{k-1}, M_T(k)] \\
&= E\{E[\tilde{x}\tilde{x}' \mid Z^k, M_T(k)] \mid Z^{k-1}, M_T(k)\} \\
&= E[P(k \mid k-1) - W(k)S(k)W(k)' \mid Z^{k-1}, M_T(k)] \\
&= P(k \mid k-1) - W(k)S(k)W(k)'. \qquad (110)
\end{aligned}$$

This proves (52). On the other hand,

$$\begin{aligned}
&E[\tilde{x}\tilde{x}' \mid Z^{k-1}, M_T(k)] \\
&\quad = \int \tilde{x}\tilde{x}' p(\tilde{x} \mid M_T)\, d\tilde{x} = \int \tilde{x}\tilde{x}' p(\overline{x} \mid M_T)\, d\overline{x} \\
&\quad = \iint (\overline{x} - W\nu)(\overline{x} - W\nu)' p(\overline{x} \mid \nu, M_T) \\
&\qquad \cdot p(\nu \mid M_T)\, d\overline{x}\, d\nu \\
&\quad = \iint (\overline{x}\overline{x}' + W\nu\nu'W' - \overline{x}\nu'W' - W\nu\overline{x}') \\
&\qquad \cdot p(\overline{x} \mid \nu, M_T) p(\nu \mid M_T)\, d\overline{x}\, d\nu \\
&\quad = \int \overline{x}\overline{x}' p(\overline{x} \mid M_T)\, d\overline{x} + W \int \nu\nu'W' \\
&\qquad \cdot p(\nu \mid M_T)\, d\nu - A' - A \\
&\quad = P(k \mid k-1, M_T) + WE[\nu\nu' \mid M_T]W' - A' - A
\end{aligned} \qquad (111)$$

where

$$\begin{aligned}
A &= W \iint \nu\overline{x}' p(\overline{x} \mid \nu, M_T) p(\nu \mid M_T)\, d\overline{x}\, d\nu \\
&= W \int \nu \int (\tilde{x} + W\nu)' p(\tilde{x} \mid \nu, M_T)\, d\overline{x}\, p(\nu \mid M_T)\, d\nu \\
&= W \int \nu \int (\tilde{x} + W\nu)' \mathcal{N}[\tilde{x}; 0, \tilde{P}]\, d\overline{x}\, p(\nu \mid M_T)\, d\nu \\
&= W \int \nu\nu'W' p(\nu \mid M_T)\, d\nu \\
&= WE[\nu\nu' \mid M_T]W'. \qquad (112)
\end{aligned}$$

Consequently, from the lemma below (Lemma 3), we have

$$\begin{aligned}
E[\mathrm{MaSE} \mid M_T] &= P(k \mid k-1, M_T) - WE[\nu\nu' \mid M_T]W' \\
&= P(k \mid k-1, M_T) \\
&\quad - \frac{c_T(k)}{P\{M_T(k)\}} W(k)S(k)W(k)' \qquad (113)
\end{aligned}$$

We now establish the following lemma.

LEMMA 3 *The covariance of the measurement residual under M_T is*

$$E[\nu\nu' \mid M_T] = \frac{c_T}{P\{M_T\}} S \qquad (114)$$

where c_T is a scalar, given by (54)*, and S was defined by* (16).

PROOF Following the same procedure as in the proof of Lemma 2, one has, by Theorem 1,

$$p(\tilde{\nu} \mid M_T) = \frac{P_D e^{-\lambda V_{\|\tilde{\nu}\|^2}}}{P\{M_T\}} \mathcal{N}[\tilde{\nu}; 0, I] U(\tilde{\nu}; \tilde{R}_\gamma) \qquad (115)$$

$$\begin{aligned}
E[\tilde{\nu}\tilde{\nu}' \mid M_T] P\{M_T\} &= P_D \int_{\tilde{R}_\gamma} \tilde{\nu}\tilde{\nu}' e^{-\lambda V_{\|\tilde{\nu}\|^2}} \mathcal{N}[\tilde{\nu}; 0, I]\, d\tilde{\nu} \\
&= c_T I \qquad (116)
\end{aligned}$$

$$c_T = \frac{P_D c_n}{2(2\pi)^{n/2}} \int_0^\gamma q^{n/2} e^{-\beta q^{n/2} - q/2}\, dq. \qquad (117)$$

□

Finally, Theorem 5 follows from this lemma and (110)–(113).

REMARK With Assumption A3 one has

$$\begin{aligned}
&\lim_{\gamma \to \infty} E\{[z_T - \hat{z}(k \mid k-1)][z_T - \hat{z}(k \mid k-1)]' \mid Z^{k-1}\} \\
&\quad = S. \qquad (118)
\end{aligned}$$

The event M_T implies that only those target-originated measurements in the gate that have a NDS smaller than those of the false measurements are selected for estimate update. As a result, it is clear that this set of specially selected target-originated measurements are closer to the gate center, and thus have a smaller covariance, than the set of target-originated measurements without such a selection. Since this selection uses an elliptic gate whose shape and orientation are determined by S, the following holds for the measurement residual of (15)

$$E[\nu\nu' \mid M_T] \le S \qquad (119)$$

where the matrix inequality is well defined in view of (114). Hence, one has, from (114),

$$c_T \le P\{M_T\}. \qquad (120)$$

Clearly, the equality holds iff $\lambda \to 0$ and $\gamma \to \infty$, i.e., iff the gating and the NN criterion has no effect at all. Alternatively, (120) can be shown directly as follows.

Let

$$\begin{aligned}
b &= P\{M_T\} - c_T \\
&= \frac{P_D}{2^{n/2}\Gamma(n/2)} \int_0^\gamma \left(1 - \frac{1}{n} q\right) \\
&\quad \times q^{n/2-1} e^{-\beta q^{n/2} - q/2}\, dq. \qquad (121)
\end{aligned}$$

It is clear that $b > 0$ (i.e., $P\{M_T\} > c_T$) for $0 < \gamma < n$ and

$$\frac{P_D}{2^{n/2}\Gamma(n/2)}\int_n^{\gamma}\left(1-\frac{1}{n}q\right)q^{n/2-1}e^{-\beta q^{n/2}-q/2}\,dq \tag{122}$$

decreases monotonically for $n < \gamma < \infty$ as γ increases. Hence, b tends to its minimum as $\gamma \to \infty$, given by

$$b_{\min} = \lim_{\gamma\to\infty} P\{M_T(k)\} - \lim_{\gamma\to\infty} c_T \geq 0. \tag{123}$$

The inequality in (123) follows from (114) and the fact

$$\lim_{\gamma\to\infty} E[\nu\nu' \mid M_T] \leq S. \tag{124}$$

□

J. Proof of Theorem 6

Theorem 6 follows directly from Theorems 4 and 5, (45), the fact that the three event probabilities sum up to unity, and the following

$$\begin{aligned}
&\sum_{i=0}^{2} E\{[x(k)-\hat{x}(k \mid k-1)] \\
&\qquad \times [x(k)-\hat{x}(k \mid k-1)]' \mid M_i\}P\{M_i(k)\} \\
&\quad = E\{[x(k)-\hat{x}(k \mid k-1)][x(k)-\hat{x}(k \mid k-1)]'\} \\
&\quad = P(k \mid k-1). \qquad (125)
\end{aligned}$$

□

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